

Topological complexity of collision free motion planning algorithms in the presence of multiple moving obstacles

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ABSTRACT. We study motion planning algorithms for collision free control of multiple objects in the presence of moving obstacles. We compute the topological complexity of algorithms solving this problem. We apply topological tools and use information about cohomology algebras of configuration spaces. The results of the paper may potentially be used in systems of automatic traffic control.

1. Introduction

The theory of robot motion planning [11], [10] has developed a broad variety of algorithms designed for different real life situations. Most of these algorithms assume that the robot performs an online search of the scene and eventually finds its way to the goal. In this approach one understands that the robot has initially a very limited knowledge of the scene and supplements this knowledge by using sensors, vision, memory and perhaps some ability to analyze these data.

A completely different situation arises when one controls simultaneously multiple objects (robots) moving in a coordinated way in a fully known environment. Assume, for instance, that we have n objects moving in \mathbf{R}^3 with no collisions and avoiding the obstacles whose geometry is prescribed in advance. In this case the dimension $3n$ of the configuration space of the system is large (if the number n of controlled objects is large) and therefore the online search algorithms as described above become much less effective.

The topological approach to the robot motion planning problem initiated in [1, 2] is applicable in some situations with high dimensional configuration space under the assumption that the configuration space of the system is known in advance. One divides the whole configuration space into pieces (local domains) and prescribes continuous motions (local rules) over each of the local domains. The minimal number of such local domains is the measure of topological complexity of the problem.

Formally, the topological complexity is a numerical invariant $\mathbf{TC}(X)$ of the configuration space of the system [1]. Its introduction was inspired by the earlier well-known work of S. Smale [16] and V. Vassiliev [17] on the theory of topological

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complexity of algorithms of solving polynomial equations. The approach of [1, 2] was also based on the general theory of robot motion planning algorithms described in the book of J.-C. Latombe [11] and on the abstract theory of a genus of a fiber space developed by A.S. Schwarz [14]. Paper [6] contains a recent survey.

Paper [5] solves the problem of finding the topological complexity of motion planning algorithms for controlling many particles moving in \mathbf{R}^3 or on the plane \mathbf{R}^2 with no collisions. Such motion planning algorithms appear in automatic traffic control problems. Paper [5] uses the techniques of the theory of subspace arrangements [12]. It was shown in [5] that the complexity is roughly twice the number of particles (it is $2n - 2$ for the planar case and $2n - 1$ for the spatial case). At the moment we do not know specific motion planning algorithms with complexity linear in n . However a quadratic in n motion planning algorithm was described in §26 of [6].

In paper [3] we studied the problem of computing the topological complexity of the motion planning problem for n particles moving on a graph with forbidden collisions. It was shown that for large numbers of particles the topological complexity depends only on the geometry of the graph and is independent of the number of particles. Some specific motion planning algorithms for this problem were also constructed [3].

In the present paper we study algorithms solving the following motion planning problem. Several objects are to be transported from an initial configuration $A_1, A_2, \dots, A_n \in \mathbf{R}^3$ to a final configuration $B_1, B_2, \dots, B_n \in \mathbf{R}^3$ such that in the process of motion there occur no collision between the objects and such that the objects do not touch the obstacles in the process of motion.

We make several simplifying assumptions: (a) Each object is represented by a single point; (b) each obstacle is represented by a single point; (c) collision between two objects occurs if they are situated at the same point in space; (d) an object touches an obstacle if the points representing the object and the obstacle coincide. We also make an important assumption that the behavior of the obstacles is known in advance. We show that because of this assumption the problem becomes topologically equivalent to the similar problem when the obstacles are stationary.

One of our results seems surprising: we show that complexity of collision free navigation of many objects in the presence of moving obstacles is essentially independent of the number of obstacles and grows linearly with the number of controlled objects.

We believe that the conclusions of this paper will remain valid in a more general and realistic situation when the objects and the obstacles are represented by small balls, possibly of different radii, and the control requirements are to avoid tangencies between objects and obstacles.

2. Statement of the result

Let us explain our notation. n denotes the number of moving objects and m the number of obstacles. The symbol $F(X, n)$ denotes the configuration space of n distinct points in a topological space X ; in other words, $F(X, n)$ is the subset of the Cartesian power $X \times X \times \dots \times X$ (n times) consisting of configurations (x_1, \dots, x_n) (where $x_i \in X$) with $x_i \neq x_j$ for $i \neq j$.

We assume that the controllable objects lie in \mathbf{R}^{r+1} where $r \geq 1$ is an integer. Clearly, the practical applications are covered by the cases $r = 1$ or $r = 2$ which

will be considered in more detail later. In this section we will assume that r is arbitrary.

The algorithms we study take as input the following data:

- (a) the initial configuration $A = (A_1, A_2, \dots, A_n) \in F(\mathbf{R}^{r+1}, n)$ of the objects, $A_i \in \mathbf{R}^{r+1}$, with $A_i \neq A_j$;
- (b) the desired final configuration of the objects $B = (B_1, B_2, \dots, B_n) \in F(\mathbf{R}^{r+1}, n)$, where $B_i \in \mathbf{R}^{r+1}$, $B_i \neq B_j$;
- (c) the trajectory of moving obstacles $C(t) = (C_1(t), C_2(t), \dots, C_m(t)) \in F(\mathbf{R}^{r+1}, m)$, $C_i(t) \neq C_j(t)$ for any $t \in [0, 1]$.

Here $t \in [0, 1]$ is the time. We assume that $A_i \neq C_j(0)$ and $B_i \neq C_j(1)$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$. This means that initially the objects are all distinct and disjoint from the obstacles at time $t = 0$ and the desired positions of the objects are pairwise distinct and distinct from positions of the obstacles at time $t = 1$.

The output of the algorithm is a continuous motion of the objects

$$\gamma_i(t) \in \mathbf{R}^{r+1}, \quad t \in [0, 1], \quad i = 1, \dots, n$$

satisfying the following conditions:

- (α) $\gamma_i(t)$ are continuous functions of t ;
- (β) $\gamma_i(0) = A_i$ and $\gamma_i(1) = B_i$;
- (γ) for any $t \in [0, 1]$ one has $\gamma_i(t) \neq \gamma_k(t)$ for $i \neq k$ where $i, k \in \{1, \dots, n\}$;
- (δ) for any $t \in [0, 1]$ one has $\gamma_i(t) \neq C_j(t)$ where $i = 1, \dots, n$ and $j = 1, \dots, m$.

Let the function

$$Y_m : [0, 1] \rightarrow F(\mathbf{R}^{r+1}, m)$$

represent the motion of the obstacles. For any time $t \in [0, 1]$ the obstacles are positioned at the points

$$Y_m(t) = (C_1(t), C_2(t), \dots, C_m(t)).$$

Next we introduce a few more notations:

$$E(Y_m) = \{\gamma : [0, 1] \rightarrow F(\mathbf{R}^{r+1}, n); \gamma(t) \cap Y_m(t) = \emptyset, \quad \forall t \in [0, 1]\},$$

$$B(Y_m) = \{(A, B) \in F(\mathbf{R}^{r+1}, n) \times F(\mathbf{R}^{r+1}, n); A \cap Y_m(0) = \emptyset, \quad B \cap Y_m(1) = \emptyset\}.$$

Elements of $E(Y_m)$ are motions of the objects avoiding the obstacles, elements of $B(Y_m)$ represent the input data: the initial configuration A and the desired configuration B .

There is a canonical map

$$(2.1) \quad \pi(Y_m) : E(Y_m) \rightarrow B(Y_m)$$

given by $\pi(Y_m)(\gamma) = (\gamma(0), \gamma(1))$. A *section* of π is a map (possibly discontinuous)

$$(2.2) \quad s : B(Y_m) \rightarrow E(Y_m),$$

satisfying $\pi(Y_m) \circ s = 1_{B(Y_m)}$. Clearly, s assigns to any input data $(A, B) \in B(Y_m)$ a motion of the controllable objects starting at configuration A , ending at B and avoiding mutual collisions between objects and between objects and the moving obstacles. We conclude that algorithms solving our motion planning problem are in one-to-one correspondence with sections of map (2.1) (we will see below that it

is a fibration). Given a section s as above, then for any pair $(A, B) \in B(Y_m)$, the output of the algorithm is $s(A, B) \in E(Y_m)$.

Hence, our task is to estimate the complexity of finding a section of $\pi(Y_m)$.

We recall that *the Schwarz genus* [14] of a continuous map $f : X \rightarrow Y$ map is the minimal integer k such that Y can be covered by k open sets $U_1 \cup U_2 \cdots \cup U_k = Y$ with the property that over each U_i there exists a continuous section $s_i : U_i \rightarrow X$, $f \circ s_i = 1_{U_i}$. We apply this notion to estimate the complexity of finding a section to (2.1).

DEFINITION 2.1. The complexity of the motion planning problem for moving n objects in the presence of m moving obstacles is defined as the Schwarz genus of map (2.1).

In practical terms this means the following. One divides the space of all possible inputs $B(Y_m)$ into several domains (called *the local domains*) and specifies a continuous motion planning algorithm (called *a local rule*) over each of the domains. The complexity of the problem is defined (according to Definition 2.1) as the minimal number of local domains in all possible algorithms of this type.

Now we state our main result:

THEOREM 2.2. *The complexity of motion planning problem of moving $n \geq 2$ objects in \mathbf{R}^3 avoiding collisions in the presence of $m \geq 1$ moving obstacles equals $2n + 1$. The complexity of the similar planar problem (i.e. when the objects are restricted to lie in the plane \mathbf{R}^2) is $2n$ if $m = 1$ and it is $2n + 1$ for $m \geq 2$.*

This theorem follows by combining Theorem 3.1 and Theorems 5.1 and 6.1 which are stated and proven below.

3. Reduction to the case of stationary obstacles

The following theorem shows that Definition 2.1 gives the notion of complexity which coincides with a special case of the notion of navigational complexity of topological spaces $\mathbf{TC}(X)$ which was studied previously [1]. Recall that $\mathbf{TC}(X)$ denotes the Schwarz genus of the path space fibration

$$(3.1) \quad \pi : X^I \rightarrow X \times X$$

where X^I denotes the space of all continuous paths $\gamma : I = [0, 1] \rightarrow X$ with the compact-open topology and $\pi(\gamma) = (\gamma(0), \gamma(1))$.

THEOREM 3.1. (a) *The map $\pi(Y_m)$ (given by (2.1)) is a locally trivial fibration;* (b) *The fiberwise homeomorphism type of fibration $\pi(Y_m)$ is independent of the trajectory of moving obstacles Y_m ; in particular, it equals the Schwarz genus of the special case of (2.1) when the obstacles are stationary;* (c) *The Schwarz genus of $\pi(Y_m)$ equals*

$$\mathbf{TC}(F(\mathbf{R}^{r+1} - S_m, n))$$

where $S_m = \{s_1, \dots, s_m\} \subset \mathbf{R}^{r+1}$ is an arbitrary fixed m -element subset.

PROOF. First we find a continuous family of homeomorphisms

$$\psi_t : \mathbf{R}^{r+1} \rightarrow \mathbf{R}^{r+1}, \quad t \in [0, 1]$$

having the properties

$$(3.2) \quad \psi_0 = \text{id}, \quad \text{and} \quad \psi_t(Y_m(t)) = Y_m(0), \quad t \in [0, 1].$$

Equation (3.2) is understood as an equality between ordered sets; in other words, (3.2) states that

$$(3.3) \quad \psi_t(C_i(t)) = C_i(0), \quad t \in [0, 1].$$

The existence of such isotopy ψ_t is a standard fact of the manifold topology; formally it follows from the well known isotopy extension theorem.

Next we consider the commutative diagram of continuous maps

$$(3.4) \quad \begin{array}{ccc} E(Y_m) & \xrightarrow{F} & X^I \\ \downarrow \pi(Y_m) & & \downarrow \pi \\ B(Y_m) & \xrightarrow{G} & X \times X \end{array}$$

where

$$X = F(\mathbf{R}^{r+1} - Y_m(0), n)$$

and π is the canonical path space fibration (3.1) for the special case $X = F(\mathbf{R}^{r+1} - Y_m(0), n)$. Maps F and G are defined as follows:

$$(3.5) \quad F(\gamma)(t) = \psi_t(\gamma(t)), \quad \gamma \in E(Y_m), \quad t \in [0, 1],$$

$$(3.6) \quad G(A, B) = (\psi_0(A), \psi_1(B)), \quad (A, B) \in B(Y_m).$$

Clearly, F and G are homeomorphisms and diagram (3.4) is commutative. This implies statements (a) - (c) of Theorem 3.1. \square

4. Cohomology of configuration space $F(\mathbf{R}^{r+1} - S_m, n)$

In this section we collect some known topological results which will be used in this paper. All cohomology groups have \mathbf{Z} as coefficients.

Let $S_m = \{s_1, \dots, s_m\} \subset \mathbf{R}^{r+1}$ be a fixed sequence of m distinct points in Euclidean space \mathbf{R}^{r+1} . Here $r \geq 1$ is an integer. The space $F(\mathbf{R}^{r+1} - S_m, n)$ represents configurations of n particles in \mathbf{R}^{r+1} avoiding mutual collisions and collisions with m stationary obstacles s_1, \dots, s_m .

The theorems we state in this section are known, they can be found in Chapter V of the book by Fadell and Husseini [7]; we have adjusted the notations as required for our needs. The space $F(\mathbf{R}^2 - S_m, n)$ (here $r = 1$) can be identified with the complement of an arrangement of affine hyperplanes in \mathbf{C}^n , so one can also use the book [12] as the reference.

We begin by noting that $F(\mathbf{R}^{r+1} - S_m, n)$ embeds in the configuration space $F(\mathbf{R}^{r+1}, n+m)$ via the map $(y_1, \dots, y_n) \mapsto (y_1, \dots, y_n, s_1, \dots, s_m)$ (in fact this embeds $F(\mathbf{R}^{r+1} - S_m, n)$ as the fibre over (s_1, \dots, s_m) of the locally trivial fibration $F(\mathbf{R}^{r+1}, n+m) \rightarrow F(\mathbf{R}^{r+1}, m)$ which projects to the last m points of the configuration). The cohomology algebra $H^*(F(\mathbf{R}^{r+1}, n+m); \mathbf{Z})$ is well known, and can be described as follows.

For each pair of integers $i \neq j$ such that $1 \leq i, j \leq n+m$, consider the map

$$\phi_{ij} : F(\mathbf{R}^{r+1}, n+m) \rightarrow S^r, \quad (y_1, y_2, \dots, y_{n+m}) \mapsto \frac{y_i - y_j}{|y_i - y_j|} \in S^r.$$

Letting $[S^r] \in H^r(S^r)$ denote the fundamental cohomology class of the sphere S^r , we obtain cohomology classes

$$e_{ij} = \phi_{ij}^*[S^r] \in H^r(F(\mathbf{R}^{r+1}, n+m); \mathbf{Z}).$$

THEOREM 4.1. ([7], Theorem V.4.2) *$H^*(F(\mathbf{R}^{r+1}, n+m))$ is the free associative graded-commutative algebra generated by the classes e_{ij} for $1 \leq i < j \leq n+m$, subject to the relations*

- (i) $e_{ij}^2 = 0$, and
- (ii) $e_{ij}e_{ik} = (e_{ij} - e_{ik})e_{jk}$ for any triple $i < j < k$.

THEOREM 4.2. ([7], V.4.2) *The homomorphism*

$$H^*(F(\mathbf{R}^{r+1}, n+m); \mathbf{Z}) \rightarrow H^*(F(\mathbf{R}^{r+1} - S_m, n); \mathbf{Z})$$

induced by inclusion is an epimorphism, with kernel equal to the ideal generated by those e_{ij} having $i > n$ and $j > n$.

COROLLARY 4.3. *In $H^*(F(\mathbf{R}^{r+1} - S_m, n); \mathbf{Z})$ there are relations*

$$e_{ij}e_{ik} = 0$$

for any triples i, j, k such that $i \leq n$ and $j, k > n$.

THEOREM 4.4. ([7], Theorem V.4.3) *A basis for $H^*(F(\mathbf{R}^{r+1} - S_m, n); \mathbf{Z})$ is given by the set of monomials*

$$e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_s j_s}$$

where $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ and for each q with $1 \leq q \leq s$ we have $i_q < j_q$ and $1 \leq j_q \leq n+m$.

We remark here that the maximum length of such a monomial is $(n-1)$ when $m = 0$, and n when $m \geq 1$. Hence the highest dimension in which the integral cohomology of $F(\mathbf{R}^{r+1} - S_m, n)$ is non-trivial is $r(n-1)$ when $m = 0$ and rn when $m \geq 1$. Note also that when $m > 1$ the given basis includes all monomials of the form

$$(4.1) \quad \mu_I = \prod_{i \in I} e_{i, n+1} \prod_{i \notin I} e_{i, n+2}$$

for every $I \subset \{1, \dots, n\}$. For $r = 1$ this fact also follows from the notion of **nb**c-basis for an appropriate ordering of the generators (see [12]).

In what follows we will specialise to the cases $r = 2$ and $r = 1$.

5. Computing $\mathbf{TC}(F(\mathbf{R}^3 - S_m, n))$

THEOREM 5.1. *One has:*

$$\mathbf{TC}(F(\mathbf{R}^3 - S_m, n)) = \begin{cases} 2n-1 & \text{if } m = 0, \\ 2n+1 & \text{if } m \geq 1. \end{cases}$$

PROOF. We first establish a lower bound. We shall use the cohomological lower bound given by Theorem 7 of [1]. Set $\bar{e}_{ij} = 1 \otimes e_{ij} - e_{ij} \otimes 1$. It is a zero-divisor of the cohomology algebra. Note that $(\bar{e}_{ij})^2 = -2(e_{ij} \otimes e_{ij}) \neq 0$. Consider the following product $\pi = \prod_{i=1}^{n-1} (\bar{e}_{in})^2$. We find $\pi = (-2)^{n-1} \mu \otimes \mu$, where $\mu = \prod_{i=1}^{n-1} e_{in}$. The monomial μ is nonzero by Theorem 4.4, and hence the product π of length $2(n-1)$ is nonzero. This gives the lower bound $\mathbf{TC}(F(\mathbf{R}^3 - S_m, n)) \geq 2n-1$. Now assuming

that $m \geq 1$ we have a nontrivial product $\prod_{i=1}^n (\bar{e}_{i(n+1)})^2$ of length $2n$, which gives the lower bound $\mathbf{TC}(F(\mathbf{R}^3 - S_m, n)) \geq 2n + 1$ when $m \geq 1$.

To obtain the upper bound, note that $F(\mathbf{R}^3 - S_m, n)$ can be viewed as the complement of a finite collection of codimension 3 affine subspaces in \mathbf{R}^{3n} , so it is simply-connected by an easy transversality argument. Since it has finitely generated torsion-free homology and cohomology, it has the homotopy type of a CW-complex consisting of one k -cell for each k -dimensional element in the basis for cohomology given by Theorem 4.4 (see Chapter 4.C of [9]). This minimal cell structure is made explicit in Theorem VI.8.2 of [7]. In particular, it has the homotopy type of a polyhedron of dimension $2(n - 1)$ when $m = 0$ and $2n$ when $m \geq 1$. We now apply Corollary 5.3 of [2] stating that for a 1-connected polyhedron Y ,

$$\mathbf{TC}(Y) \leq \dim(Y) + 1,$$

which together with homotopy invariance of \mathbf{TC} completes the proof. \square

6. Computing $\mathbf{TC}(F(\mathbf{R}^2 - S_m, n))$

THEOREM 6.1. *One has:*

$$\mathbf{TC}(F(\mathbf{R}^2 - S_m, n)) = \begin{cases} 2n - 2 & \text{if } m = 0, \\ 2n & \text{if } m = 1, \\ 2n + 1 & \text{if } m \geq 2. \end{cases}$$

PROOF. The statement for the first two cases follows immediately from [5] and the fact that $F(\mathbf{R}^2 - X_1, n)$ is homotopy equivalent to $F(\mathbf{R}^2, n + 1)$ (see [7], p.15).

In what follows we assume that $m \geq 1$. We first establish a lower bound using again the cohomological lower bound given by Theorem 7 from [1]. Set as above $\bar{e}_{ij} = 1 \otimes e_{ij} - e_{ij} \otimes 1$ and consider the product

$$\pi = \prod_{i=1, \dots, n, j=n+1, n+2} \bar{e}_{ij}.$$

It is clear that π can be expressed as a linear combination of pure tensors $\mu_1 \otimes \mu_2$ where μ_i are monomials in the e_{ij} complementary to each other in π . Since the highest non-zero dimension in $H^*(F(\mathbf{R}^2 - S_m, n); \mathbf{Z})$ is n both μ_i should have degree n in order for $\mu_1 \otimes \mu_2$ not to vanish. Also it follows from the relation of Corollary 4.3 that the nonvanishing summands of π are of the form $\mu_I \otimes \mu_{\bar{I}}$, where μ_I is the monomial defined by (4.1), I runs over all subsets of $\{1, 2, \dots, n\}$ and \bar{I} denotes the complement of I . Since the set $\{\mu_I | I \subset \{1, \dots, n\}\}$ is a subset of a basis of $H^n(F(\mathbf{R}^2 - S_m, n); \mathbf{Z})$ given by Theorem 4.4, no cancellations are possible, whence $\pi \neq 0$. This gives the inequality $\mathbf{TC}(F(\mathbf{R}^2 - S_m, n)) \geq 2n + 1$.

The opposite inequality follows immediately from Theorem 5.2 of [2] since $F(\mathbf{R}^2 - S_m, n)$ has the homotopy type of a connected polyhedron of dimension n (see [12]). This completes the proof of the Theorem. \square

7. Concluding remarks

First we note that our main result Theorem 2.2 follows by simply combining Theorems 3.1, 5.1 and 6.1.

Let us compare the following two control problems: (1) motion planning for moving n objects in \mathbf{R}^3 with no collisions and avoiding collisions with $m \geq 1$ moving obstacles; and (2) motion planning for moving n objects in \mathbf{R}^3 with the

only requirement that they avoid a single point obstacle (in particular, the objects are allowed to collide, i.e. to occupy the same position in space). Note that the integer m in problem (1) can be arbitrarily large. According to Theorem 2.2 the topological complexity of the problem (1) is $2n + 1$. Surprisingly, problem (2) also has complexity $2n + 1$. Indeed, the configuration space of problem (2) is Cartesian power of n copies of $\mathbf{R}^3 - \{0\}$ which is homotopy equivalent to

$$S^2 \times S^2 \times \cdots \times S^2 \quad (n \text{ times})$$

which has topological complexity $2n + 1$ as it is easy to see. Hence, surprisingly problem (1) which is intuitively more “complicated” has the same topological complexity as problem (2).

This comparison shows that in general the notion of topological complexity is only a partial reflection of real difficulty of a motion planning problem.

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